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Explicit auto-transformations of integrable chains

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Abstract. A construction scheme for explicit auto-transformations of integrable discrete-differential equations (chains) is presented. These transformations are rather convenient to obtain the exact solutions for chains as well as associated partial differential systems. On the other hand they exemplify new integrable discrete mappings. Their group properties are also of great interest. The scheme is illustrated by several examples of integrable systems which contains the nonlinear Schrödinger system and the Landau–Lifshits model.

1. Introduction

Bäcklund auto-transformations for integrable PDEs are known to be very often evolution discrete-differential equations ([1, 2]; [3] was one of the first where the Toda and Volterra models were interpreted as Bäcklund transformations for integrable systems of the Schrödinger type). For example, the nonlinear Schrödinger system

$$u_t = u_{xx} + 2u^2 v \qquad -v_t = v_{xx} + 2v^2 u \tag{1}$$

is related to two chains:

$$q_{jxx} = \exp(q_{j+1} - q_j) - \exp(q_j - q_{j-1}) \tag{2}$$

$$u_{jx} = -u_{j+1} - u_j^2 v_{j+1} \qquad v_{jx} = v_{j-1} + v_j^2 u_{j-1} \tag{3}$$

where $j \in \mathbb{Z}$. In [1] there is a rather large list of integrable chains which consists of two essentially different classes. Chains from the first class containing the Toda lattice (2) are remarkable owing to the explicit auto-transformations for corresponding Schrödinger-type systems which they specify (see also [3]). These transformations are handy for the construction of exact solutions [4]. For example, the Toda lattice gives rise to the invertible differential substitution

$$\tilde{u} = u_{xx} - u^{-1} u_x^2 + u^2 v \qquad \tilde{v} = u^{-1}.$$

This formula allows one to construct easily a new solution $\tilde{u}$, $\tilde{v}$ of the nonlinear Schrödinger system (1), starting from its arbitrary solution u, v.

We shall be interested in the second class. Its simplest representative is the chain (3); the others can be written in the form

$$\begin{cases} u_{jx} = [\alpha u_j v_{j+1} + \beta(u_j + v_{j+1}) + \gamma](u_{j+1} - u_j) \\ v_{jx} = [\alpha v_j u_{j-1} + \beta(v_j + u_{j-1}) + \gamma](v_j - v_{j-1}) \end{cases} \tag{4}$$

where α, β, γ are arbitrary constants, or in the form

$$\begin{cases} u_{jx}=r_j/(u_{j+1}+v_{j+1})-1/2\,\partial r_j/\partial v_{j+1}, \\ v_{jx}=-r_{j-1}/(u_{j-1}+v_{j-1})+1/2\,\partial r_{j-1}/\partial u_{j-1} \end{cases} \tag{5}$$

where r_j is a polynomial with constant coefficients:

$$r_j=r(u_j, v_{j+1})=\alpha u_j^2 v_{j+1}^2+\beta u_j v_{j+1}(u_j-v_{j+1}) \\ +\gamma u_j v_{j+1}+\delta(u_j-v_{j+1})^2+\varepsilon(u_j-v_{j+1})+\mu. \tag{6}$$

It seemed for some time that the chains (3), (4) and (5) were not so convenient for the construction of exact solutions as chains similar to (2). We intend to demonstrate that this is not so. It turns out that explicit auto-transformations arise in this case as well, but at the next discretization level.

We make our words more exact by example of the chain (3). One can obtain a non-trivial generalization of (3) introducing additional parameters β_j:

$$\begin{cases} u_{jx}=-u_{j+1}-\beta_j u_j-u_j^2 v_{j+1} \\ v_{jx}=v_{j-1}+\beta_{j-1}v_j+v_j^2 u_{j-1}. \end{cases} \tag{7}$$

The chain (7), in contrast to (3), admits an explicit auto-transformation B_k which is given by

$$B_k: \begin{cases} \tilde{u}_k=u_k+(\beta_{k-1}-\beta_k)\dfrac{u_{k-1}}{1-u_{k-1}v_{k+1}} \\ \tilde{v}_k=v_k+(\beta_k-\beta_{k-1})\dfrac{v_{k+1}}{1-u_{k-1}v_{k+1}} \\ \tilde{\beta}_{k-1}=\beta_k \qquad \tilde{\beta}_k=\beta_{k-1} \end{cases} \tag{8}$$

in the kth node and is identical in others ($\tilde{u}_j=u_j$, $\tilde{v}_j=v_j$ for $j\neq k$, $\tilde{\beta}_j=\beta_j$ for $j\neq k-1, k$). Note that (8) is not an auto-transformation in the strict sense of the word, since it permutes parameters of the chain. For this reason it is sometimes useful to consider instead of a single chain, the whole set obtained from it by permutations of parameters.

The transformations (8) allow one to construct exact solutions of the chain (7) and, at the same time, of the nonlinear Schrödinger system (1), starting from some appropriate initial solution. No problems arise when we construct solutions for higher symmetries of (1) as well as solutions admitting the scalar or complex reduction. The scheme of the construction of multi-soliton solutions is given in section 3.

As in the discussed example, we shall present integrable generalizations for the chains (4) and (5) containing additional parameters β_j, and auto-transformations suitable for the integration of associated Schrödinger type systems. The associated systems are of the form

$$u_t=u_{xx}+f(u, v, u_x) \qquad v_t=-v_{xx}+g(u, v, v_x)$$

and represent key equations from the complete list of Schrödinger-type integrable systems obtained in [5] with the help of the symmetry approach. To achieve our objective we use the linked zero curvature representations for the chains and systems associated with them. It should be pointed out that we shall not consider the important problem of construction of such representations; all representations in this paper were found directly from the determining equation (9). The general scheme is stated and substantiated in section 2. Results regarding the chains (4) and (5) are enumerated in section 5.

Note that every chain (3), (4), (5) is connected with two associated systems at least [1, 7]. This means that the transformations presented permit one to solve many more systems than those described in this paper. The chains (4), (5) do not exchaust examples of chains analogous to (3). Multi-field generalizations of the chain (3) have been obtained in [8]. The scheme presented is applicable to other types of integrable systems (cf [9] where the KdV equation is considered).

The transformations discussed are of interest also due to their group properties defined by the following identities

$$B_j^2 = (B_j B_{j+1})^3 = 1 \qquad B_i B_j = B_j B_i \qquad i \neq j \pm 1.$$

Additionally, they provide new examples of integrable discrete mappings which are actively being investigated at present ([9–13]). The problem connected with the transformations (8) is to investigate the dynamics of u_j, v_j under the action of the group generated by B_k. In other words, the integration problem for multi-valued mapping (or correspondence) is raised. When the additional periodicity condition is imposed, this problem becomes closely connected with the theory of the finite-band integration of the associated system (1). This connection is illustrated in section 4.

2. General scheme

Let us consider a chain admitting zero curvature representation

$$(W_j)_x = U_{j+1} W_j - W_j\, U_j \qquad j \in \mathbb{Z} \tag{9}$$

where U_j, W_j are 2×2 matrices, $U_j = U(\lambda, u_j, v_j)$, $W_j = W(\lambda, u_j, v_{j+1}, \beta_j)$, λ is a spectral parameter, and β_j are parameters of the chain. In the examples below tr $U_j = 0$ and the formula $(\det W)_x = \det W \operatorname{tr}(W_x W^{-1})$ implies that $\det W_j = \delta(\lambda, \beta_j)$ does not depend on x.

We define a transformation B_k by the relations

$$B_k : \tilde{W}_k \tilde{W}_{k-1} = W_k W_{k-1} \qquad \tilde{W}_j = W_j \qquad j \neq k, k-1 \tag{10}$$

where $\tilde{W}_j = W(\lambda, \tilde{u}_j, \tilde{v}_{j+1}, \tilde{\beta}_j)$. These relations give a system of algebraic equations for $\tilde{u}_{k-1}, \tilde{v}_k, \tilde{u}_k, \tilde{v}_{k+1}, \tilde{\beta}_{k-1}, \tilde{b}_k$. As a rule this system is overdetermined. However, it is consistent, for it always has an identical solution. It should be remarked that sometimes no other solutions exist. For example, our scheme does not give results in the case of the Toda and Volterra models, although these equations are well known to admit Bäcklund auto-transformations [17–21]. Nevertheless, the class of the chains admitting a non-trivial transformation seems to be large enough. For the chains (4), (5) the transformations (10) are of the specific form

$$B_k : \begin{cases} \tilde{u}_k = P(u_{k-1}, u_k, v_{k+1}, \beta_{k-1}, \beta_k) & \tilde{u}_j = u_j \qquad j \neq k \\ \tilde{v}_k = Q(u_{k-1}, v_k, v_{k+1}, \beta_{k-1}, \beta_k) & \tilde{v}_j = v_j \qquad j \neq k \\ \tilde{\beta}_{k-1} = \beta_k, \tilde{\beta}_k = \beta_{k-1} \qquad \tilde{\beta}_j = \beta_j & j \neq k-1, k \end{cases} \tag{11}$$

where P, Q are rational functions (see section 5). The rule about the change of β_j can be derived from the equality $\det W_k W_{k-1} = \det \tilde{W}_k \tilde{W}_{k-1}$. The fact that P and Q are rational follows from the overdeterminedness of the system (10), which allows us to reduce it to a linear one. In formulae like (11) below we shall write down only actually transformed variables for short.

Theorem 1 explains why transformations of the form (10), (11) do not change the chains under consideration.

Theorem 1. Let all the derivatives u_{jx}, v_{jx} be uniquely determined not only by the system (9) but also by

$$\begin{aligned} (W_j)_x &= U_{j+1}W_j - W_jU_j \qquad j \neq k, k-1 \\ (W_kW_{k-1})_x &= U_{k+1}(W_kW_{k-1}) - (W_kW_{k-1})U_{k-1}. \end{aligned} \tag{12}$$

Let a transformation (10) be of the form (11). Then the chain (9) is invariant under this transformation up to a change $\beta_k \leftrightarrow \beta_{k-1}$.

Proof. The system (12) follows from (9). In accordance with theorem 1 hypotheses, (9) and (12) are equivalent. After the transformation (10), (11), the system (12) remains a system of the same form, but with $\tilde{u}_j$, $\tilde{v}_j$, $\tilde{\beta}_j$ in place of u_j, v_j, β_j. Thus the chain (9) remains unchanged up to a change $\beta_k \leftrightarrow \beta_{k-1}$. □

The hypotheses of theorem 1 express some rigidity of the chain (9). The following condition on the matrices W_j is stronger.

Condition (A) *If*

$$\tilde{W}_{k+p} \ldots \tilde{W}_k = W_{k+p} \ldots W_k \tag{13}$$

then $\tilde{\beta}_{k+p} = \beta_{\sigma(k+p)}, \ldots, \tilde{\beta}_k = \beta_{\sigma(k)}$, *where* σ *is some permutation. If* σ *is identical, then* $\tilde{W}_{k+p} = W_{k+p}, \ldots, \tilde{W}_k = W_k$.

Additionally, we assume (and it is natural) that the equalities $\tilde{W}_j = W_j$ for all j imply that $\tilde{\beta}_j = \beta_j$, $\tilde{u}_j = u_j$, $\tilde{v}_j = v_j$ for all j. In particular the first of the theorem 1 hypotheses follows from condition (A), since the equation (10) possesses only one solution if the permutation σ is fixed.

Theorem 2. Let the chain (9) satisfy condition (A) and admit non-trivial transformations (10). Then the identities

$$B_j^2 = (B_jB_{j+1})^3 = 1 \qquad B_iB_j = B_jB_i \qquad i \neq j \pm 1 \tag{14}$$

are valid, defining the code of the group G generated by B_j. Any transformation satisfying (13) (and of course $\tilde{W}_j = W_j, j \neq k, \ldots, k+p$) belongs to this group.

Proof. Each of the transformations B_j^2, $(B_jB_{j+1})^3$, $(B_jB_i)^2$, $i \neq j \pm 1$, satisifies a relation of the form (13) and acts on β_j identically. By virtue of condition (A) they act identically on the variables u_j, v_j as well.

Any transformation (13) specifies some permutation on the set of β_j. There exists a composition of this transformation with some element of the group G which gives identical permutation. This composition satisfies one of the relations (13) and therefore is identical transformation. □

If a chain admits non-trivial transformations (10), then according to theorem 2 any attempt to generalize them with the help of (13) fails: every transformation obtained will be their composition. Nevertheless the situation is possible when the transformations (10) are trivial, and then one may try to obtain a non-trivial transformation by (13) with $p>1$. We present such an example in section 4.

The partial differential system associated with the chain (9) is given by zero curvature representation

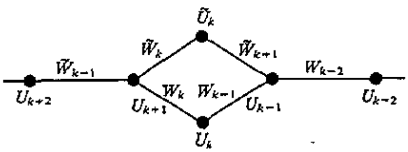
$$U_t = V_x + [V, U] \tag{15}$$

where the matrix $U = U(\lambda, u, v)$ coincides with the matrix U from the representation (9). In order to construct the system (15) solutions by the transformations (10), it is convenient to pass to the representation

$$(W_j)_t = V_{j+1} W_j - W_j V_j \tag{16}$$

where $V_j = V(\lambda, \beta_j, u_j, v_j, u_{jx}, v_{jx}, \ldots)$. Since this chain is of the same type as (9), it is invariant under the transformations B_j too (see theorem 1). The representation (15) is the compatibility condition for the chains (9) and (16). So one can construct solutions of (15) with the help of B_j, generating common solutions of the chains (9) and (16). This takes place for higher symmetries of (15) as well.

Generally speaking, the transformation (10) may not be related with any chain (9), and also the chain (9) may not be related with any system (15). In this paper we consider the case when there are both discretization levels. In this case the chain (9) defines an infinite sequence of Bäcklund transformations for the system (15). The formula (10) expresses the commutability of two Bäcklund transformations, as one can see from the diagram



Here the lower branch corresponds to the original chain, and the upper one corresponds to the transformed chain.

The so-called nonlinear superposition principle for Bäcklund transformations is well known [6]. If there is a matrix potential U_{k-1} and two potentials U_k and $\tilde U_k$ related to U_{k-1} by Bäcklund transformation, then this principle allows one to obtain (in an algebraic way) a new potential U_{k+1} which is a result of double Bäcklund transformation. The relationships (10) often can be rewritten as nonlinear superposition formulae. However, there exist examples when this is impossible. We also remark that Bäcklund transformations for the Painléve equations can be derived from the transformations (10) of suitable chains [9].

3. Nonlinear Schrödinger system

The system (1) and chain (7) can be written in the form (15), (9) with

$$U = \begin{pmatrix} \lambda & v \\ -u & -\lambda \end{pmatrix} \qquad V = -2\lambda U - \begin{pmatrix} uv & v_x \\ u_x & -uv \end{pmatrix} \qquad W_j = \begin{pmatrix} 1 & -v_{j+1} \\ u_j & 2\lambda - u_j v_{j+1} - \beta_j \end{pmatrix}.$$

It is an easy exercise to prove that (10) yields the transformations (8).

Theorem 3. The transformations (8) act on the set of chains (7) and satisfy the identities (14).

Proof. It is sufficient to check that the condition (A) holds. This is evident for the first part of (A), since det $W_j=2\lambda-\beta_j$. If $2\lambda=\beta_k$ we obtain that ker $W_{k+p}\dots W_k$ is spanned on the vector $(v_{k+1},1)^T$, whereby $\tilde{v}_{k+1}=v_{k+1}$ (see (13)). Further, one can easily verify that

$$W_{k+p}\dots W_{k+1}=\begin{pmatrix} * & * \\ \alpha & (2\lambda)^p+\dots \end{pmatrix}$$

where deg $\alpha<p$, and the sign $*$ designates the inessentials. So

$$W_{k+p}\dots W_{k+1}W_k=\begin{pmatrix} * & * \\ (2\lambda)^p u_k+\dots & * \end{pmatrix}$$

and therefore $\tilde{u}_k=u_k$. We obtain $\tilde{W}_k=W_k$, and the proof is reduced to the case of the lesser number of matrices in (13). □

Let us demonstrate how the transformations (8) help to construct system (1) solutions. According to the previous section these transformations allow one to generate common solutions of the equations (9) and (16). The matrix equation (16) represents the chain

$$u_{jt}=-u_{j+1,x}-(u_jv_{j+1}+\beta_j)u_{jx}+u_{j+1}u_jv_{j+1}+u_j^2v_j$$
$$-v_{jt}=v_{j-1,x}+(v_ju_{j-1}+\beta_{j-1})v_{jx}+v_{j-1}v_ju_{j-1}+v_j^2u_j.$$

One can easily check that, by virtue of (7), this chain is equivalent to the sequence of *the nonlinear Schrödinger systems*

$$u_{jt}=u_{jxx}+2u_j^2v_j \qquad -v_{jt}=v_{jxx}+2v_j^2u_j. \tag{17}$$

Thus the transformations (8) are fit for the construction of common solutions of the chains (7) and (17). This means one can construct solutions of (7) with u_j, v_j satisfying, for all j, the Schrödinger system (1), starting from some trivial solution of this kind. This statement remains valid for higher symmetries of (1) as well, but we shall not prove it.

First of all we have to construct some initial solution of (7). There exist several possibilities to do this, and the simplest way is to impose the conditions

$$u_0=u_1=u_2=\dots=0 \qquad v_0=v_{-1}=v_{-2}=\dots=0$$

which split (7) in two linear chains

$$-u_{-jx}=\beta_{-j}u_{-j}+u_{-j+1} \qquad v_{jx}=\beta_{j-1}v_j+v_{j-1} \qquad j>0.$$

The system (1) gives the set of heat equations

$$u_{-jt}=u_{-jxx} \qquad -v_{jt}=v_{jxx} \qquad j>0$$

which permit one to find easily the dependence on t. In a similar way, the next symmetry of (1)

$$u_{t_3}=u_{xxx}+6uvu_x \qquad v_{t_3}=v_{xxx}+6vuv_x$$

is turned into linear equations again, and in general

$$u_{-jt_n}=\partial^n u_{-j}/\partial x^n \qquad v_{jt_n}=(-1)^{n+1}\partial^n v_j/\partial x^n \qquad j>0,$$

where $t_2=t$. Assuming that all β_j are different, we find

$$u_{-1}=\exp(y_{-1}) \qquad u_{-2}=\exp(y_{-1})/(\beta_{-1}-\beta_{-2})+\exp(y_{-2}) \qquad \ldots$$

$$v_1=\exp(-y_0) \qquad v_2=\exp(-y_0)/(\beta_0-\beta_1)+\exp(-y_1) \qquad \ldots$$

where $y_j=c_j-\beta_j x+\beta_j^2 t-\beta_j^3 t_3+\ldots$.

Applying transformations (8) to the solutions obtained one can construct new solutions of (17). Consider, for example, transformation $T=\ldots B_k\ldots B_1 B_0$, which is, obviously, a correctly defined transformation of the chain. The jth component of the new solution can be expressed through the components of the old solution in explicit form as a finite continued fraction:

$$T(u_j)=u_j+\frac{\beta_j-\beta_{-1}}{v_{j+1}}\,\frac{1}{-u_{j-1}}+\frac{\beta_{j-1}-\beta_{-1}}{v_j}\,-\ldots+\frac{\beta_0-\beta_{-1}}{v_1}\,\frac{1}{-u_{-1}}$$

$$T(v_j)=v_j+\frac{\beta_j-\beta_{-1}}{1/v_{j+1}-T(u_{j-1})} \qquad j\geqslant 0 \qquad T(\beta_j)=\beta_{j+1} \qquad j\geqslant -1.$$

Note that the chain (7) admits two reductions:

$$v_j=(-1)^j u_{-j} \qquad \beta_j=-\beta_{-j-1} \tag{18}$$

and, after the change $x=\mathrm{i}\xi$

$$v_j=\bar{u}_{-j} \qquad \beta_j=\bar{\beta}_{-j-1}. \tag{19}$$

It is easy to see that $B_0, B_{-1}B_1, B_{-2}B_2, \ldots$ preserve these reductions. Moreover, the component (u_0, v_0) satisfies the condition $u_0=v_0$ in the first case and the condition $v_0=\bar{u}_0$ in the second one. Choosing the constants c_j in the proper way, one can easily obtain an initial solution satisfying the conditions (18) or (19). Starting from this one can construct solutions of the MKdV equation

$$u_{t_3}=u_{xxx}+6u^2u_x$$

or nonlinear Schrödinger equation

$$\mathrm{i}u_\tau=u_{\xi\xi}-2|u|^2u \qquad (t=\mathrm{i}\tau)$$

respectively. Consider, for example, the case of MKdV. Note that the reduction (18), unlike (19), is compatible only with odd flows, which implies $y_j=c_j-\beta_j x-\beta_j^3 t_3-\ldots$. Transformation $T=\ldots B_k B_{-k}\ldots B_1 B_{-1} B_0$ acts as follows (we are interested only in variables with non-negative indices):

$$T(\beta_j)=\beta_{j+1} \qquad j\geqslant 0 \qquad T(u_0)=u_0+\frac{2\beta_0}{v_1+1/v_1}$$

$$T(u_j)=u_j+\frac{\beta_j+\beta_0}{v_{j+1}}\,\frac{1}{-u_{j-1}}+\frac{\beta_{j-1}+\beta_0}{v_j}\,-\ldots+\frac{2\beta_0}{v_1}\,\frac{1}{+v_1}$$

$$T(v_j)=v_j+\frac{\beta_j+\beta_0}{1/v_{j+1}-T(u_{j-1})} \qquad j\geqslant 1.$$

The MKdV solution $u=T^n(u_0)$ obtained by iteration of this transform depends on $2n$ constants $\beta_0, c_0, \ldots, \beta_{n-1}, c_{n-1}$. Let us rewrite it in the form

$$u=\frac{2\beta_0}{v_1+1/v_1}+\frac{2\beta_1}{T(v_1)+1/T(v_1)}+\ldots+\frac{2\beta_{n-1}}{T^{n-1}(v_1)+1/T^{n-1}(v_1)}$$

then assuming that all β_j and $\exp(c_j)$ are real, we see that u is regular for all real values of x and t and therefore it performs the general n-soliton solution of MKdV.

4. Examples of integrable discrete mappings

A quite natural generalization of the Liouville integrability notion for the discrete mappings is given in [12]. Following this paper we call a correspondence (i.e. in general multi-valued mapping) $\Phi: M \to M$ symplectic if it preserves symplectic structure on M. A function on M being preserved under the action of Φ is called the first integral or invariant. A symplectic correspondence Φ is called integrable if it admits $n=\frac{1}{2}\dim M$ functionally independent involutory first integrals. The discrete version of the Liouville theorem states that if the common level surface of the invariants of an integrable correspondence Φ is compact, then it is diffeomorphic to a disconnected union of n-dimensional tori, and Φ defines a multi-valued shift on it.

The transformations (10) give a new wide class of integrable correspondences. We shall show this by example of the transformations (8) first. Consider the system obtained from (7) by imposing the periodicity condition

$$u_{j+N}=u_j \qquad v_{j+N}=v_j \qquad \beta_{j+N}=\beta_j \qquad j\in\mathbb{Z}. \tag{20}$$

It is assumed in this section that indices in all formulae belong to $\mathbb{Z}_N$. The dynamical system obtained defines the finite-band solutions of (1) depending on t as on an integration constant [1, 14]. Let us consider the integration problem for the N-valued correspondence B defined by the transformations $B_1, \ldots, B_N$. The matrix $\hat{W}_j=W_{j+N-1}\ldots W_j$ satisfies the equation

$$(\hat{W}_j)_x=[U_j, \hat{W}_j]$$

which implies that tr $\hat{W}_j$ is a generating function for the first integrals of (7) and (20). The algebraic curve

$$\Gamma: \det\{\xi I-\hat{W}_j\}=0$$

is obviously preserved too. On the other hand, it is also clear that Γ is preserved under the action of the correspondence B.

After imposing the periodicity condition (20) the chain (7) becomes an integrable in the Liouville-sense Hamiltonian system with Poisson bracket

$$\{v_i, u_j\}=\delta_{i,j+1} \qquad \{v_i, v_j\}=\{u_i, u_j\}=0$$

and the Hamiltonian

$$H=\sum_1^N h_j$$

where

$$h_j=u_jv_j+\beta_ju_jv_{j+1}+u_j^2v_{j+1}^2/2. \tag{21}$$

One can check that tr $\hat{W}_j$ provides exactly N functionally independent first integrals in involution.

It is easy to verify that the transformations (8) are Poisson mappings, i.e. they preserve the bracket: $\{\tilde{v}_i, \tilde{u}_j\} = \{v_i, u_j\}$, etc. Since the dynamics of β_j is trivial, it is convenient to pass from the transformations B_j to their combinations leaving β_j unchanged. Note that it follows from (14) that the group G generated by B_j is isomorphic to the affine Weyl group $\tilde{A}_{N-1}$. The subgroup acting on β_j identically is generated by

$$T_j = (B_j \ldots B_{j+N-1})^{N-1}.$$

Each of the transformations T_j is a Poisson mapping not changing the system (7), (20) and therefore is integrable in the above sense. Thus the original correspondence B is a combination of an N-valued integrable correspondence and the group of permutations of N elements.

It is well known that the explicit linearization of the system (1) and its higher symmetries, that is the transition to the action-angle variables, is realized on the curve Γ Jacoby manifold. The commutability of the transformations (8) and the dynamics with respect to x and all times immediately implies that transformation T_j corresponds to the shift by some constant vector on the Jacoby manifold. So the dynamics is quite trivial, and it is remarkable only that this shift can be described in terms of the given system by the explicit formulae (8).

It is clear from what has been said above how to use the transformations (8) for numerical investigation of the system (7), (20). Indeed, it seems that the iterations of one of the transformations T_j must give the phase portrait of the system. However, numerical simulations show that the transformations (8) are bad illustrations of the discrete Liouville theorem. The level surfaces are not compact, there are no Liouville tori, and we fail to obtain the whole phase portrait. The reason is quite obvious. The system admits a reduction by means of the introduction of new variables

$$p_j = u_j v_{j+1} \qquad q_j = u_{j+1}/u_j. \tag{22}$$

If the functions p_j, q_j are known, the solution u_j, v_j is found by the integration:

$$-u_{jx}/u_j = q_j + p_j + \beta_j \qquad v_{jx}/v_j = p_{j-1} + q_{j-2}p_{j-2}/p_{j-1} + \beta_{j-1}.$$

For $N=2$ it is easy to prove that p_j, q_j are elliptic functions, and therefore u_j, v_j generally grow or vanish exponentially, in accordance with numerical experiments.

So we see it is convenient to pass to the variables (22). Our second example of integrable mapping deals with just this case. It turns out that the chain (7) as well as the system (1) and the transformations (8) can be rewritten in terms of (22). In fact the change (22) is equivalent to the Bäcklund transformation

$$uv = p_x + pq \qquad -u_x/u = p + q + \beta$$

from (1) into

$$q_t = q_{xx} - (q^2 + 2pq + 2\beta q)_x \qquad p_t = -p_{xx} - (p^2 + 2pq + 2\beta p)_x. \tag{23}$$

In terms of p_j, q_j the chain (7) looks as follows:

$$\begin{cases} p_{jx} = p_{j-1}q_{j-1} - p_j q_j \\ q_{jx} = q_j(q_j + p_j + \beta_j - q_{j+1} - p_{j+1} - \beta_{j+1}). \end{cases} \tag{24}$$

The transformation B_k takes the form

$$\tilde{p}_{k-1}=p_{k-1}-\frac{\alpha p_k}{p_k-q_{k-1}} \qquad \tilde{p}_k=p_k\left(1+\frac{\alpha}{p_k-q_{k-1}}\right)$$
$$\tilde{q}_{k-1}=q_{k-1}\left(1+\frac{\alpha}{p_k-q_{k-1}}\right) \qquad \tilde{q}_k=q_k\left(1-\frac{\alpha}{p_k-q_{k-1}+\alpha}\right) \tag{25}$$
$$\tilde{\beta}_{k-1}=\beta_k \qquad \tilde{\beta}_k=\beta_{k-1}$$

where $\alpha=\beta_k-\beta_{k-1}$. Although this system and chain do not belong to the main class

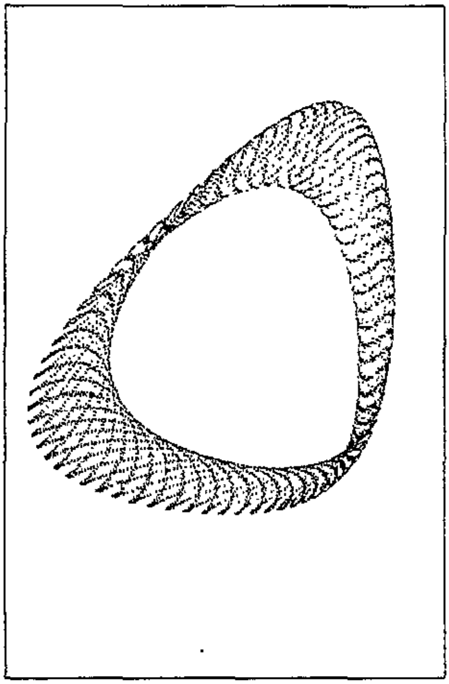


Figure 1. Liouville torus.

which is discussed in this paper, we wish to consider this didactic example in detail. The chain (24) is Hamiltonian with Poisson bracket:

$$\{p_j, q_j\}=-q_j \qquad \{p_{j+1}, q_j\}=q_j$$

(the others vanish) and Hamiltonian density:

$$h_j=p_j^2/2+\beta_j p_j+p_j q_j.$$

Note that the new Poisson structure is degenerate with Casimir function $J=q_1\ldots q_N$. The transformations B_j are Poisson ones and preserve J. As before the periodicity condition reduces the chain (24) to a Liouville integrable system and transformations $T_j=(B_j\ldots B_{j+N-1})^{N-1}$ where B_k is given by (25) become integrable mappings. The pictures below correspond to $N=3$ and represent the projections on the plane (p_1, q_1) of the images of a random initial vector under action of transform T_1 iterations. The level surface can be non-compact again, but now without asymptotic tendency to infinity. In the compact case the level surface is diffeomorphic to an N-1-dimensional torus

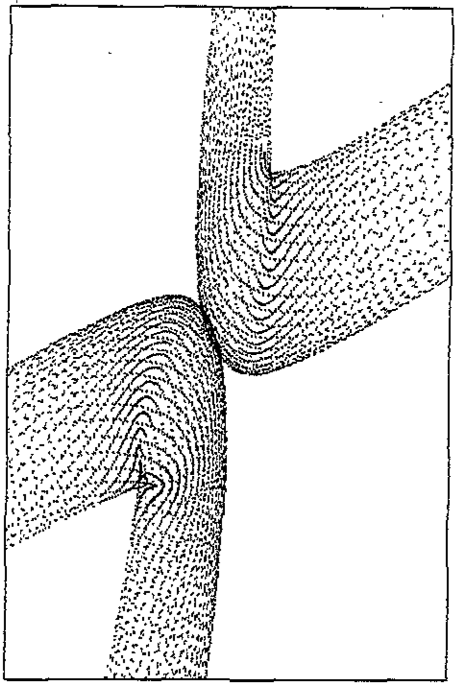


Figure 2. Non-compact level surface.

dotted in a quite regular way. As a whole the picture looks like a uniform winding on torus and provides the visual demonstration of discrete Liouville theorem.

The system (23) was considered in [3, 5]. The chain (24) (without β_j) is closely connected with the relativistic Toda lattice [22]. It should be remarked that in this case, in contrast to the nonlinear Schrödinger system, the associated system contains the parameter β_j. The chain (24) defines not an auto-transformation, but the transformation of the system (23) with $\beta=\beta_j$ into (23) with $\beta=\beta_{j+1}$. Zero curvature representations are given by

$$U=\begin{pmatrix} \lambda-s & p \\ -q & -\lambda+s \end{pmatrix}$$

$$V=-2(\lambda+s)U+\begin{pmatrix} (p-q)/2 & -p \\ -q & (q-p)/2 \end{pmatrix}_x$$

$$W_j=q_j^{-1/2}\begin{pmatrix} q_j & -p_{j+1} \\ q_j & 2\lambda-\beta_{j+1}-p_{j+1} \end{pmatrix}$$

where $s=(p+q+\beta)/2$. It is easy to prove that the equation (10) with the given matrix W_j possesses only the identical solution, and therefore no non-trivial transformations arise. It turns out we have to refactorize the product of three matrices in order to obtain the formulae (25): the transformation B_k is defined by

$$B_k: \qquad \tilde{W}_k\tilde{W}_{k-1}\tilde{W}_{k-2}=W_kW_{k-1}W_{k-2} \qquad \tilde{W}_j=W_j \qquad j\neq k, k-1, k-2.$$

So we see that sometimes the general scheme from section 2 needs modifications.

5. Landau–Lifshits model and other examples

The results obtained for the nonlinear Schrödinger system can be generalized for integrable systems considered in this section. All the chains below possess a Hamiltonian structure

$$\begin{pmatrix} u_j \\ v_{j+1} \end{pmatrix}_x = \Delta_j \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta h_j/\delta u_j \\ \delta h_j/\delta v_{j+1} \end{pmatrix} \tag{26}$$

where

$$\delta h_j/\delta u_j = \sum_k \partial h_k/\partial u_j$$

(cf [1]). Note that the factor $\Delta_j = \Delta(u_j, v_{j+1})$ can be eliminated by a point transformation, but this makes the form of the chains more complicated. For the chain (7) $\Delta_j = -1$ and h_j is of the form (21). The structure functions Δ_j and the Hamiltonian densities h_j for the other chains are given below.

For each chain under consideration we present the associated partial differential system and the transformation (10). Matrices specifying the representations (9) and (15) are given as well.

Example 1. The system

$$u_t = u_{xx} + (2uv + \beta)u_x \qquad -v_t = v_{xx} - (2vu + \beta)v_x \tag{27}$$

at $\beta = 0$ was discussed in [15]. The zero curvature representation (15) is given by the matrices

$$U = \begin{pmatrix} r & \lambda u \\ \lambda v & -r \end{pmatrix} \qquad V = (2r + \beta)U + \begin{pmatrix} (vu_x - uv_x)/2 & \lambda u_x \\ -\lambda v_x & (uv_x - vu_x)/2 \end{pmatrix}$$

where $r = (uv - \lambda^2)/2$, and

$$W_j = (u_j v_{j+1} + \beta_j)^{-1/2} \begin{pmatrix} u_j v_{j+1} + \beta_j - \lambda^2 & \lambda u_j \\ \lambda v_{j+1} & \beta_j \end{pmatrix}.$$

The chain (9) takes the form

$$\begin{cases} u_{jx} = (u_j v_{j+1} + \beta_j)(u_{j+1} - u_j) \\ v_{jx} = (v_j u_{j-1} + \beta_{j-1})(v_j - v_{j-1}) \end{cases}$$

and can be written in the Hamiltonian form (26) with

$$\Delta_j = u_j v_{j+1} + \beta_j \qquad h_j = (u_{j+1} - u_j)v_{j+1}.$$

The transformation (10) is given by

$$B_k:\begin{cases}\tilde{u}_k=u_k+(\beta_k-\beta_{k-1})\dfrac{u_{k-1}-u_k}{\beta_k+u_{k-1}v_{k+1}}\\[2mm] \tilde{v}_k=v_k+(\beta_{k-1}-\beta_k)\dfrac{v_{k+1}-v_k}{\beta_{k-1}+u_{k-1}v_{k+1}}\\[2mm] \tilde{\beta}_{k-1}=\beta_k \qquad \tilde{\beta}_k=\beta_{k-1}.\end{cases}$$

As in the case of (23) there is a changing parameter in the system (27).

Example 2. The system

$$u_t=u_{xx}+2(u+v)u_x \qquad -v_t=v_{xx}-2(v+u)v_x$$

is equivalent to the Kaup system [16]. The matrices U, V and W_j are

$$U=\begin{pmatrix}(u-v)/2 & (u+\lambda)(v+\lambda)\\ 1 & (v-u)/2\end{pmatrix}$$

$$V=(u+v-2\lambda)U+\begin{pmatrix}(u_x+v_x)/2 & \lambda(u_x-v_x)+vu_x-uv_x\\ 0 & -(u_x+v_x)/2\end{pmatrix}$$

$$W_j=(u_j+v_{j+1})^{-1/2}\begin{pmatrix}u_j-\lambda & u_jv_{j+1}+(\lambda-\beta_j)(u_j+v_{j+1})+\lambda^2\\ 1 & v_{j+1}-\lambda\end{pmatrix}$$

and the chain (9) is of the form

$$\begin{cases}u_{jx}=(u_j+v_{j+1})(u_{j+1}-u_j+\beta_j)\\ v_{jx}=(v_j+u_{j-1})(v_j-v_{j-1}-\beta_{j-1}).\end{cases}$$

Its Hamiltonian structure (26) is defined by the functions

$$\Delta_j=u_j+v_{j+1} \qquad h_j=(u_{j+1}-u_j)v_{j+1}+\beta_j(u_j+v_{j+1})$$

and its transformation (10) is given by the formula

$$B_k:\begin{cases}\tilde{u}_k=u_k+(\beta_{k-1}-\beta_k)\dfrac{u_k+v_{k+1}}{u_{k-1}+v_{k+1}-\beta_{k-1}}\\[2mm] \tilde{v}_k=v_k+(\beta_k-\beta_{k-1})\dfrac{v_k+u_{k-1}}{u_{k-1}+v_{k+1}-\beta_k}\\[2mm] \tilde{\beta}_{k-1}=\beta_k \qquad \tilde{\beta}_k=\beta_{k-1}.\end{cases}$$

The rest of the examples deals with systems (15) of the form

$$\begin{cases}u_t=u_{xx}-\dfrac{2}{u+v}(u_x^2+P(u))+\dfrac{1}{2}P'(u)\\[2mm] -v_t=v_{xx}-\dfrac{2}{u+v}(u_x^2+P(-v))-\dfrac{1}{2}P'(-v)\end{cases} \tag{28}$$

where the degree of the polynomial P is less than or equal to 4. It is well known [1] that the stereographic projection

$$S_1=\frac{1+uv}{u+v} \qquad S_2=-\mathrm{i}\frac{1-uv}{u+v} \qquad S_3=\frac{u-v}{u+v}$$

and substitution $t=\mathrm{i}\tau$ bring the Landau–Lifshits model

$$S_\tau=S\times S_{xx}+S\times JS \qquad S\in\mathbb{R}^3 \qquad \langle S,S\rangle=1 \qquad J=\mathrm{diag}(J_1,J_2,J_3)$$

to (28) with $P(u)=\varepsilon u^4+\delta u^2+\varepsilon$, $2\delta=J_1+J_2-2J_3$, $4\varepsilon=J_2-J_1$. One obtains the isotropic Heisenberg model case at $P=0$ and anisotropic one at $\varepsilon=0$ or $\delta=\pm 2\varepsilon$.

The linear fractional transformations

$$\tilde{u}=\frac{au+b}{cu+d} \qquad \tilde{v}=\frac{-av+b}{cv-d}$$

do not change the form of system (28). The polynomial P is changed as in the equation $u_x^2=P(u)$. This observation allows us to reduce the system invsetigation to the following three cases: $P(u)=\varepsilon$, $P(u)=\delta u^2$, and $P(u)=u^3+au+b$ (P may have multiple zeros in the last case).

A chain (9) corresponding to the system (28) has the form (5). It can be written in the Hamiltonian form (26) with

$$\Delta_j=r_j \qquad h_j=\ln(u_j+v_j)-\tfrac{1}{2}\ln r_j.$$

We shall specify r_j in each of the cases.

Example 3. Let us consider the case $P(u)=\varepsilon$ first. The representations (9) and (15) are given by

$$U=\frac{\lambda}{u+v}\begin{pmatrix}(u-v)/2 & uv-\varepsilon/\lambda^2\\ 1 & (v-u)/2\end{pmatrix}$$

$$V=-\lambda U+\frac{\lambda}{(u+v)^2}\begin{pmatrix}(uv)_x & v^2u_x-u^2v_x+\varepsilon(u_x-v_x)/\lambda^2-2\varepsilon(u+v)/\lambda\\ v_x-u_x & -(uv)_x\end{pmatrix}$$

$$W_j=r_j^{-1/2}\begin{pmatrix}\lambda v_{j+1}-\beta_j(u_j+v_{j+1}) & -\lambda u_jv_{j+1}+\varepsilon/\lambda-\varepsilon/\beta_j\\ -\lambda & \lambda u_j-\beta_j(u_j+v_{j+1})\end{pmatrix}$$

where

$$r_j=-\beta_j(u_j+v_{j+1})^2-\varepsilon/\beta_j$$

and the transformation (10) is

$$B_k:\begin{cases}\tilde{u}_k=u_k+(\beta_k-\beta_{k-1})\dfrac{(u_k+v_{k+1})(u_{k-1}-u_k)-\varepsilon/\beta_k\beta_{k-1}}{\beta_k(u_k+v_{k+1})+\beta_{k-1}(u_{k-1}-u_k)}\\[2ex] \tilde{v}_k=v_k+(\beta_{k-1}-\beta_k)\dfrac{(v_k+u_{k-1})(v_{k+1}-v_k)-\varepsilon/\beta_k\beta_{k-1}}{\beta_{k-1}(v_k+u_{k-1})+\beta_k(v_{k+1}-v_k)}\\[2ex] \tilde{\beta}_{k-1}=\beta_k \qquad \tilde{\beta}_k=\beta_{k-1}.\end{cases}$$

When $\varepsilon=0$ all the formulae remain valid and correspond to the isotropic Heisenberg model.

Example 4. The anisotropic ferromagnet corresponds to the case $P(u)=\delta u^2$. There are zero curvature representations (9) and (15) with

$$U=\frac{1}{u+v}\begin{pmatrix}\lambda(u-v)/2 & uv\\ \lambda^2+\delta & \lambda(v-u)/2\end{pmatrix}$$

$$V=-\lambda U+\frac{1}{(u+v)^2}\begin{pmatrix}\lambda(uv)_x+\delta(v^2-u^2)/2 & v^2u_x-u^2v_x\\ (\lambda^2+\delta)(v_x-u_x) & -\lambda(uv)_x-\delta(v^2-u^2)/2\end{pmatrix}$$

$$W_j=r_j^{-1/2}\begin{pmatrix}\lambda v_{j+1}-\beta_jv_{j+1}+\gamma_ju_j & -u_jv_{j+1}\\ -\lambda^2-\delta & \lambda u_j-\beta_ju_j+\gamma_jv_{j+1}\end{pmatrix}$$

where

$$r_j=\gamma_ju_j^2-2\beta_ju_jv_{j+1}+\gamma_jv_{j+1}^2 \qquad \gamma_j^2-\beta_j^2=\delta$$

and the following transformation of the chain:

$$B_k:\begin{cases}\tilde{u}_k=\dfrac{(\beta_{k-1}-\beta_k)u_{k-1}v_{k+1}+u_k(\gamma_ku_{k-1}+\gamma_{k-1}v_{k+1})}{(\beta_{k-1}-\beta_k)u_k+\gamma_{k-1}u_{k-1}+\gamma_kv_{k+1}}\\ \tilde{v}_k=\dfrac{(\beta_k-\beta_{k-1})u_{k-1}v_{k+1}+v_k(\gamma_ku_{k-1}+\gamma_{k-1}v_{k+1})}{(\beta_k-\beta_{k-1})v_k+\gamma_{k-1}u_{k-1}+\gamma_kv_{k+1}}\\ \tilde{\beta}_{k-1}=\beta_k \qquad \tilde{\beta}_k=\beta_{k-1} \qquad \tilde{\gamma}_{k-1}=\gamma_k \qquad \tilde{\gamma}_k=\gamma_{k-1}.\end{cases}$$

If $\delta=0$ and $\gamma_j=-\beta_j$ we obtain the isotropic Heisenberg model case again.

Example 5. In the case of the general position a linear fractional transformation turns the polynomial P into $P(u)=u^3+au+b$. For this P, matrices defining the representation (15) of the system (28) are

$$U=\frac{1}{u+v}\begin{pmatrix}\mu & uv-\lambda(u-v)/2-\lambda^2-a\\ (u-v)/2-\lambda & -\mu\end{pmatrix} \qquad V=\frac{1}{(u+v)^2}\begin{pmatrix}h & e\\ f & -h\end{pmatrix}$$

where

$$e=(v^2-\lambda v+\lambda^2+a)u_x-(u^2+\lambda u+\lambda^2+a)v_x+\mu(u+v)(u-v+\lambda)$$

$$f=(v+\lambda)u_x+(u-\lambda)v_x+\mu(u+v)$$

$$h=\mu(v_x-u_x)+(u+v)(uv+\lambda(u-v)+2\lambda^2+a)/2.$$

The matrix W_j has the form

$$W_j=r_j^{-1/2}\begin{pmatrix}A & B\\ C & D\end{pmatrix}$$

where

$$A=(\mu+\beta_j)s_j+(\lambda-\gamma_j)(\lambda+\gamma_j-v_{j+1})$$

$$B=(\mu+\beta_j)(\lambda+u_j-v_{j+1})-(\lambda-\gamma_j)(\lambda+2\gamma_j)s_j-2\beta_j(\lambda-\gamma_j)$$

$$C=\mu+\beta_j-(\lambda-\gamma_j)s_j$$

$$D = -(\mu+\beta_j)s_j - (\lambda-\gamma_j)(\lambda+\gamma_j+u_j)$$
$$r_j = 2\beta_j(s_j^2 + u_j - v_{j+1} + \gamma_j)$$
$$s_j = (u_j v_{j+1} + \gamma_j(u_j - v_{j+1}) + a + 2\gamma_j^2)/2\beta_j.$$

The parameters μ and λ, β_j and γ_j are constrained by

$$\mu^2 + P(\lambda) = 0 \qquad \beta_j^2 + P(\gamma_j) = 0.$$

The transformation (10) is of the form

$$\tilde{u}_k = \frac{Ku_k - L}{Mu_k + N} \qquad \tilde{v}_k = \frac{Kv_k + L}{-Mv_k + N}$$

where

$$K - N = 2c^2 u_{k-1} v_{k+1} - (ac^1 + c^3)(u_{k-1} - v_{k+1}) - 2ac^2 - 4bc^1$$
$$K + N = (u_{k-1} + v_{k+1})[\gamma_k \gamma_{k-1}(ac^0 + 3c^2) + 4bc^1 + 3ac^2 + c^4]/(\gamma_{k-1} - \gamma_k)$$
$$L = c^3 u_{k-1} v_{k+1} + (ac^2 + 2bc^1)(u_{k-1} - v_{k+1}) + 4bc^2 - a^2 c^1$$
$$M = c^1 u_{k-1} v_{k+1} + c^2(u_{k-1} - v_{k+1}) - c^3$$

and by c^s we denote $c^s = \beta_k \gamma_{k-1}^{s-1} + \beta_{k-1} \gamma_k^{s-1}$.

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